

Fourier Analysis of the Parametric Resonance in Neutrino Oscillations

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Abstract

Parametric enhancement of the appearance probability of the neutrino oscillation under the inhomogeneous matter is studied. Fourier expansion of the matter density profile leads to a simple resonance condition and manifests that each Fourier mode modifies the energy spectrum of oscillation probability at around the corresponding energy; below the MSW resonance energy, a large-scale variation modifies the spectrum in high energies while a small-scale one does in low energies. In contrast to the simple parametric resonance, the enhancement of the oscillation probability is itself an slow oscillation as demonstrated by a numerical analysis with a single Fourier mode of the matter density. We derive an analytic solution to the evolution equation on the resonance energy, including the expression of frequency of the slow oscillation.

Key words: Neutrino oscillation, Parametric resonance, Fourier analysis

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Experiments over the last decade have established the oscillation among the neutrino flavors and narrowed the allowed value of oscillation parameters. Experiments of the next generation aiming for precision measurements are actively discussed, and many of them involve a neutrino oscillation under a

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very long baseline penetrating through the Earth. The matter density on such a path varies spatially, reflecting the interior structure of the Earth. The neutrino oscillation under the varying density of matter has been widely studied in works including Refs. [1, 2, 3, 4], and the oscillation probability is found to be significantly enhanced in some cases. This enhancement can be understood in a context of the parametric resonance [2, 3]. These studies usually model the density profile by the step function segmented into three pieces, among which the middle piece has a larger value than the both ends, and reproduces the qualitative feature of the numerical results [3]. In this paper, we present an alternative formulation based on a Fourier expansion of the matter density profile, which generalizes the sinusoidal matter profile studied in one of the earliest studies [2]. This approach organizes the effects of the matter profile on the neutrino oscillation by their frequencies, in terms of which the condition for the parametric resonance to take place is given.

We will analyze the neutrino oscillation in the two-flavor framework rather than the standard three-flavor one to make the discussion simple and clear. We briefly explain that this simplification does not wipe the effect of the matter density profile off the oscillation probability. Figure 1 shows the $\nu_\mu \rightarrow \nu_e$ appearance probability for the baseline length $L = 12\,000$ km. The curves (i) and (ii) are calculated in the three-flavor framework. The curve (i) assumes the matter density $\rho(x)$ that varies according to the Preliminary Reference Earth Model (PREM) [5], while (ii) assumes the constant density of 7.58 g/cm^3 , which is the average value of the PREM profile. Other values of parameters taken here are $\delta m_{21}^2 = 7.9 \times 10^{-5}\text{ eV}^2$, $\delta m_{31}^2 = 2.5 \times 10^{-3}\text{ eV}^2$, $\sin^2 2\theta_{12} = 0.84$, $\sin^2 2\theta_{23} = 1.0$, $\sin^2 2\theta_{13} = 0.05$, and $\delta_{\text{CP}} = 0$, in the standard notations of Ref. [6]. We find qualitative difference between (i) and (ii), showing a major effect of the inhomogeneity of the matter density. The curves (iii) and (iv) are obtained from the two-flavor calculation renormalized by $\sin^2 \theta_{23}$. The curve (iii) assumes the cosine density profile of the matter as $[7.58 - 4.32 \cos(2\pi x/L)]\text{ g/cm}^3$, where the two values are taken from the first two Fourier coefficients of the PREM profile. The curve (iv) assumes the constant density of 7.58 g/cm^3 . The mass-square difference and the mixing angle are taken in common as $\delta m^2 = 2.5 \times 10^{-3}\text{ eV}^2$ and $\sin^2 2\theta = 0.05$. These two curves again show the significant qualitative difference from each other. Meanwhile, we find the curves (i) and (iii) are quite similar except in the low energies, and so are (ii) and (iv). This suggests the following two points: first, the two-generation analysis is sufficient to analyze the essence of the matter profile effect; secondly, the Fourier analysis of the

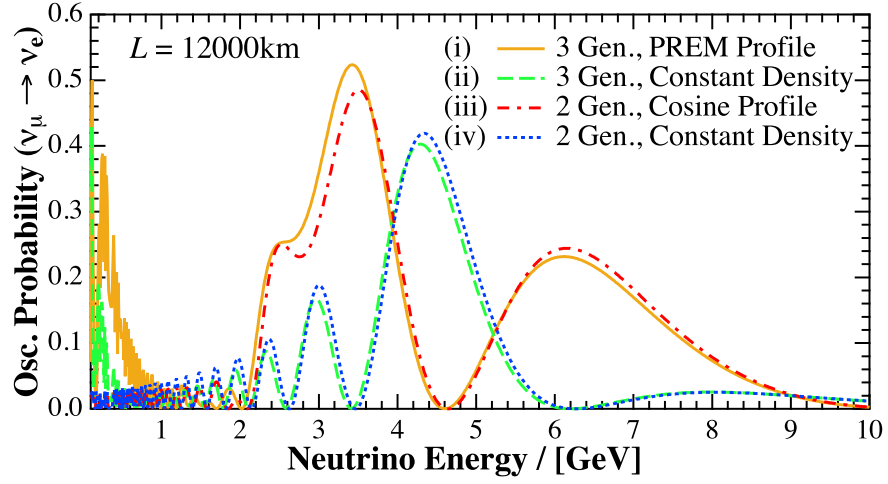


Figure 1: Oscillation probability $P(\nu_\mu \rightarrow \nu_e)$ for $L = 12000$ km. The probabilities (i) and (ii) are evaluated by three-generation analysis and (iii) and (iv) by two-generation analysis. The matter density profile used for (i) is the PREM profile [5], for (ii) and (iv) is the constant profile with 7.58 g/cm^3 , for (iii) is the profile with $[7.58 - 4.32 \cos(2\pi x/L)] \text{ g/cm}^3$. The values of other parameters taken are described in the text.

matter density profile is effective in analyzing its effect.

We study the parametric resonance of the two-flavor neutrino oscillation from ν_μ to ν_e . The evolution equation in matter is given by

$$i \frac{d}{dx} \begin{pmatrix} \nu_e(x) \\ \nu_\mu(x) \end{pmatrix} = H(x) \begin{pmatrix} \nu_e(x) \\ \nu_\mu(x) \end{pmatrix}, \quad (1a)$$

$$H(x) = \frac{1}{4E} \begin{pmatrix} -\delta m^2 \cos 2\theta + 2a(x) & \delta m^2 \sin 2\theta \\ \delta m^2 \sin 2\theta & \delta m^2 \cos 2\theta \end{pmatrix}, \quad (1b)$$

where E is the energy of neutrinos and $a(x) = 2\sqrt{2}EG_F N_A Y_e \rho(x)/[1.0 \text{ g/mol}]$ accounts for the matter effect. Here G_F , N_A , and Y_e are the Fermi constant, the Avogadro constant, and the proton-to-nucleon ratio, respectively. We reduce Eq. (1) to a differential equation of $\nu_e(\xi)$ in a dimensionless form as

$$\begin{aligned} & \nu_e''(\xi) + i\Delta_m(\xi)\nu_e'(\xi) \\ & + \left[\frac{\Delta^2}{4} - \frac{\Delta \cos 2\theta}{2} \Delta_m(\xi) + i\Delta_m'(\xi) \right] \nu_e(\xi) = 0, \end{aligned} \quad (2)$$

where $\xi \equiv x/L$ with baseline length L , $\Delta_m(\xi) \equiv a(\xi)L/2E$, and $\Delta \equiv \delta m^2 L/2E$. We eliminate the first-derivative term using

$$z(\xi) = \nu_e(\xi) \exp \left[\frac{1}{2} \int_0^\xi ds i\Delta_m(s) \right], \quad (3)$$

and then expand the matter density $\rho(\xi)$ into a Fourier series as

$$\rho(\xi) = \rho_0 + \sum_{n=1}^{\infty} \rho_n \cos 2n\pi\xi, \quad (4)$$

where ρ_0 is the average density [1]. Accordingly we expand

$$\Delta_m(\xi) = \Delta_{m0} + \sum_{n=1}^{\infty} \Delta_{mn} \cos 2n\pi\xi \quad (5)$$

to obtain

$$z''(\xi) + \left[\omega^2 + \sum_{n=1}^{\infty} u_n(\xi) + \sum_{n=2}^{\infty} \sum_{l=1}^{n-1} v_{nl}(\xi) \right] z(\xi) = 0, \quad (6a)$$

which we read as an equation of a harmonic oscillator, where

$$\omega^2 = \frac{1}{4} \left[(\Delta_{m0} - \Delta \cos 2\theta)^2 + \Delta^2 \sin^2 2\theta \right] + D \quad (6b)$$

with $D = \sum_{n=1}^{\infty} \Delta_{mn}^2/8$ gives the natural frequency of the system, while

$$u_n(\xi) = \alpha_n \cos 2n\pi\xi - i\beta_n \sin 2n\pi\xi + \gamma_n \cos 4n\pi\xi, \quad (6c)$$

$$v_{nl}(\xi) = \frac{\Delta_{mn}\Delta_{ml}}{4} [\cos 2(n+l)\pi\xi + \cos 2(n-l)\pi\xi], \quad (6d)$$

with

$$\alpha_n = \Delta_{mn}(\Delta_{m0} - \Delta \cos 2\theta)/2, \quad (7a)$$

$$\beta_n = n\pi\Delta_{mn}, \quad (7b)$$

$$\gamma_n = \Delta_{mn}^2/8 \quad (7c)$$

give oscillating potentials. The initial conditions $\nu_e(0) = 0$ and $\nu_\mu(0) = 1$ become $z(0) = 0$ and $z'(0) = -i(\Delta/2) \sin 2\theta$.

We recall that a harmonic oscillator with an oscillating potential is led to a parametric resonance when the frequency of the oscillating potential is, in the simplest case, twice the natural frequency of the system [7]. Applying this resonance condition to our system, we find that the n -th Fourier mode of the matter profile, whose frequency is $2n\pi$, gives rise to the parametric resonance of the neutrino oscillation when $2n\pi = 2\omega$. This resonance condition reads $E = E_n^{(\pm)}$ with

$$E_n^{(\pm)} = \frac{\delta m^2 L/2}{\Delta_{m0} \cos 2\theta \pm \sqrt{4n^2\pi^2 - \Delta_{m0}^2 \sin^2 2\theta - D}}, \quad (8)$$

where $E_n^{(+)}$ lies below the MSW resonance while $E_n^{(-)}$ lies above. The energy spectrum of the appearance probability at this energy hits the local minimum, which we call n -th dip, as is evident from the initial condition $\nu_e(0) = 0$ and the condition $\omega = n\pi$. To summarize, each Fourier mode of the matter density is expected to resonate the appearance probability of neutrinos at around the n -th dip of the energy spectrum.

We elucidate the effect of the n -th Fourier mode by considering a simplified model where the matter density profile consists only of the n -th mode

around the average density. The matter effect in this case is described by $\Delta_m(\xi) = \Delta_{m0} + \Delta_{mn} \cos 2n\pi\xi$, and Eq. (6) is reduced to

$$z''(\xi) + (\omega_0^2 + \alpha_n \cos 2n\pi\xi - i\beta_n \sin 2n\pi\xi + \gamma_n \cos 4n\pi\xi)z(\xi) = 0, \quad (9a)$$

$$\omega_0^2 = \frac{1}{4} \left[(\Delta_{m0} - \Delta \cos 2\theta)^2 + \Delta^2 \sin^2 2\theta + \frac{\Delta_{mn}^2}{2} \right], \quad (9b)$$

which is free from the v_{nl} terms. This is Hill's equation, which is a generalization of Mathieu's equation that appears in typical parametric resonance problems [8]. To verify the discussion in the previous paragraph, we numerically solve Eq. (9) and show in Fig. 2 the oscillation probability $P(\nu_\mu \rightarrow \nu_e)$ for the various values of ρ_1 in (a), ρ_2 in (b), and ρ_3 in (c). The average density is fixed to $\rho_0 = 7.58 \text{ g/cm}^3$ and other parameters are taken same as for (iii) and (iv) in Fig. 1. The graphs indicate that the first, the second, and the third Fourier mode of the matter density profile indeed modify the appearance probability around the first, the second, and the third dip, at which ω_0 equals to π , 2π , and 3π , respectively.

We further confirm explicitly the presence of the resonant behavior. For simplicity, we take the energy of the neutrino at the resonance energy $E_n^{(+)}$ below the MSW resonance, while the following analysis is equally applicable to the case of $E = E_n^{(-)}$. We consider the propagation of neutrinos through a repetitive density profile and calculate $P(\nu_\mu \rightarrow \nu_e)$ as a function of ξ up to $\xi > 1$ so that the resonant behavior becomes evident. The result is shown in Fig. 3, where (a) $\rho_1 = -0.3 \text{ g/cm}^3$, (b) $\rho_1 = -1.0 \text{ g/cm}^3$, and (c) $\rho_2 = -0.3 \text{ g/cm}^3$. The energy of neutrino is accordingly fixed at $E_1^{(+)}$ in (a) and (b) and $E_2^{(+)}$ in (c), while other parameters are taken common and same as in Fig. 2. The dotted curves, evaluated by the analytic formula of Eq. (15), will be discussed later. We find that the repetitive matter profile indeed raises the oscillation probability as expected from the parametric enhancement. Unlike the resonance in the classical dynamics, however, the unitarity bound prevents the probability from the divergence. The enhanced probability turns to decrease after reaching its peak, and thereby lies on a slow oscillation so that the probability consists of the fast and the slow oscillation.

We derive an analytic formula of this parametric-enhanced oscillation. We still keep $E = E_n^{(+)}$ here so that $\omega_0 = n\pi$ and $\Delta = \delta m^2 L / 2E_n^{(+)}$, and solve Eq. (9) as

$$z(\xi) = A(\xi) \cos n\pi\xi + B(\xi) \sin n\pi\xi, \quad (10)$$

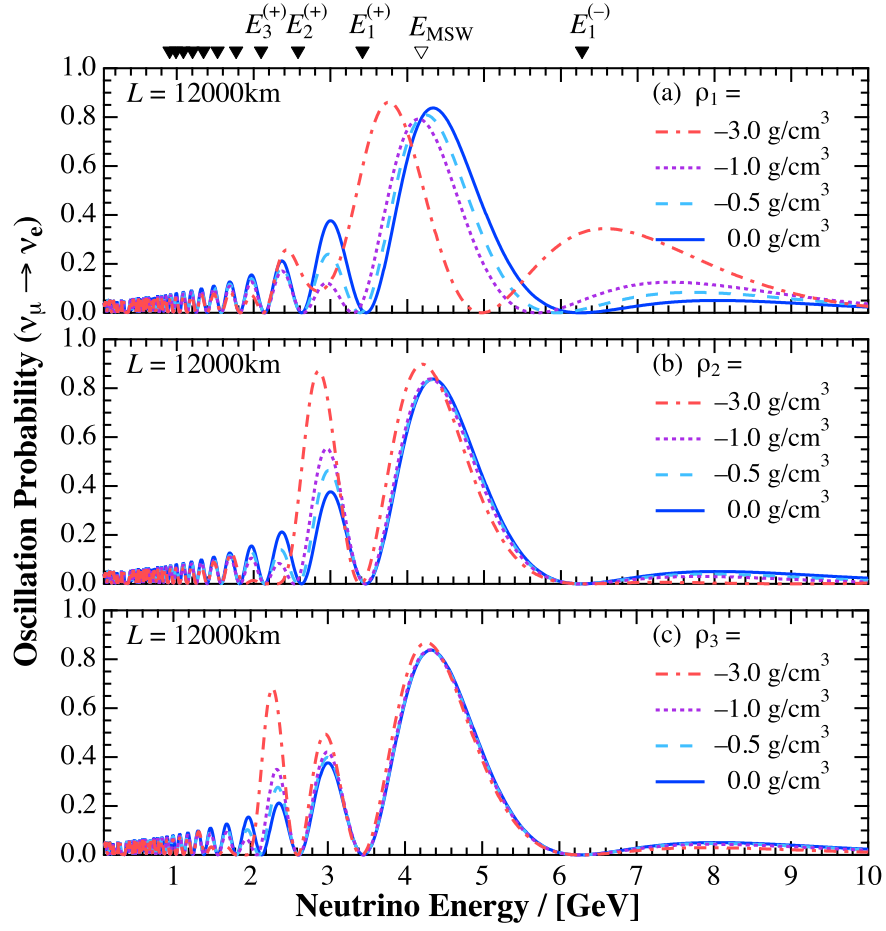


Figure 2: Oscillation probability $P(\nu_\mu \rightarrow \nu_e)$ for $\rho_n = 0.0 \text{ g/cm}^3, -0.5 \text{ g/cm}^3, -1.0 \text{ g/cm}^3$, and -3.0 g/cm^3 with (a) $n = 1$, (b) $n = 2$ and (c) $n = 3$. The average density is fixed to $\rho_0 = 7.58 \text{ g/cm}^3$ and other parameters are taken same as for (iii) and (iv) in Fig. 1.

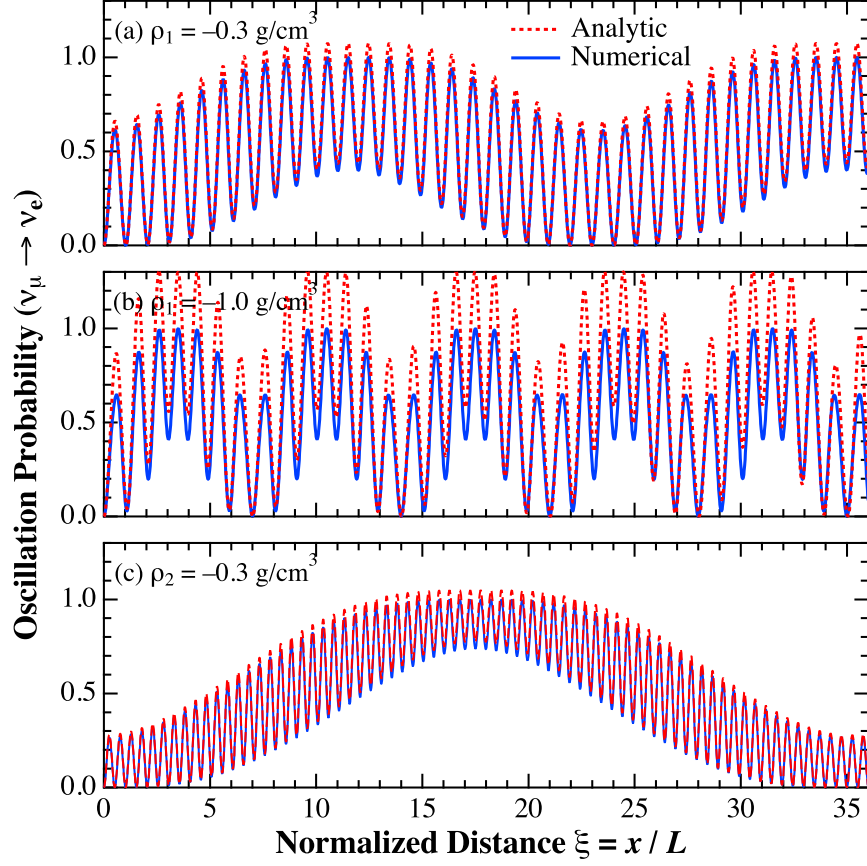


Figure 3: Oscillation probability $P(\nu_\mu \rightarrow \nu_e)$ beyond $\xi = 1$. The solid curve is given by the numerical calculation and the dotted one by using the formula of Eq. (15). The matter profile consists of $\rho_1 = -0.3 \text{ g/cm}^3$ in (a), $\rho_1 = -1.0 \text{ g/cm}^3$ in (b), and $\rho_2 = -0.3 \text{ g/cm}^3$ in (c). Neutrino energy is fixed at $E_1^{(+)}$ in (a) and (b) and $E_2^{(+)}$ in (c). The values of other parameters are same as in Fig. 2.

where $A(\xi)$ and $B(\xi)$ depend weakly on ξ . We then have

$$\begin{aligned} & \left[A''(\xi) + 2n\pi B'(\xi) + \frac{\alpha_n}{2} A(\xi) - \frac{i\beta_n}{2} B(\xi) \right] \cos n\pi\xi \\ & + \left[B''(\xi) - 2n\pi A'(\xi) - \frac{\alpha_n}{2} B(\xi) - \frac{i\beta_n}{2} A(\xi) \right] \sin n\pi\xi \\ & + (\text{faster-oscillating terms}) = 0. \end{aligned} \quad (11)$$

We neglect the faster-oscillating terms and assume $A(\xi), B(\xi) \propto e^{i\Omega\xi}$ in accordance with Floquet's theorem [9] to obtain

$$\left(\Omega^2 - \frac{1}{2}\alpha_n \right) A(\xi) - i \left(2n\pi\Omega - \frac{1}{2}\beta_n \right) B(\xi) = 0, \quad (12a)$$

$$i \left(2n\pi\Omega + \frac{1}{2}\beta_n \right) A(\xi) + \left(\Omega^2 + \frac{1}{2}\alpha_n \right) B(\xi) = 0. \quad (12b)$$

Hence Ω is given by

$$\Omega = \pm \sqrt{2}n\pi \left[1 - \sqrt{1 - \frac{\beta_n^2 - \alpha_n^2}{(2n\pi)^4}} \right]^{1/2}, \quad (13)$$

which is real since $\omega_0 = n\pi = \beta_n/\Delta_{mn}$ and $\beta_n^2 - \alpha_n^2 = \Delta_{mn}^2(2\Delta^2 \sin^2 2\theta + \Delta_{mn}^2)/8 > 0$. Thus $A(\xi)$ and $B(\xi)$ oscillate with the frequency Ω , and give $z(\xi)$ the slow oscillation mentioned above. The oscillation probability can be enhanced by this slow oscillation while it does not diverge due to its unitarity, in contrast to usual parametric resonances. When $\sqrt{\beta_n^2 - \alpha_n^2}/(4n\pi) \ll 1$, we obtain

$$|\Omega| \simeq \frac{\sqrt{\beta_n^2 - \alpha_n^2}}{4n\pi} = \frac{\Delta_{mn}}{8n\pi} \sqrt{\Delta^2 \sin^2 2\theta + \frac{1}{2}\Delta_{mn}^2}. \quad (14)$$

Taking the lowest order in Δ_{mn} of Eq. (14), we reproduce Ω_n in Ref. [2] with $2\bar{\omega} = n\omega$ and $n = 1$.

The solution of Eq. (9) with the initial conditions $\nu_e(0) = 0$ and $\nu_\mu(0) = 1$ is expressed in terms of the oscillations with the frequencies $\omega_0 = n\pi$ and Ω as

$$\begin{aligned} z(\xi) = & \frac{i\Delta \sin 2\theta}{2\Omega(4n^2\pi^2 - \alpha_n - 2\Omega^2)} \left[-i\beta_n \sin \Omega\xi \sin n\pi\xi \right. \\ & \left. + (\alpha_n + 2\Omega^2) \sin \Omega\xi \cos n\pi\xi - 4n\pi\Omega \cos \Omega\xi \sin n\pi\xi \right]. \end{aligned} \quad (15)$$

We calculate $|z(\xi)|^2$ from Eq. (15) and plot in Fig. 3 by the dotted curves. There we see that the frequency of the slow oscillation becomes larger for smaller n and larger $|\rho_n|$, as we can read from Eq. (14). The frequencies of the oscillation, both of the fast and slow ones, agree well with the numerical results. The amplitude of the slow oscillation agrees well with the numerical result in (a) and (c), while the disagreement between the two grows as the frequency Ω becomes large in (b).

The slow oscillation is extracted by averaging out the fast oscillation with the frequency $n\pi$ as

$$\begin{aligned} \langle |z(\xi)|^2 \rangle &= \frac{\Delta^2 \sin^2 2\theta}{8\Omega^2(4n^2\pi^2 - \alpha_n - 2\Omega^2)^2} \times \\ &\quad \left\{ [(\alpha_n + 2\Omega^2)^2 + \beta_n^2] \sin^2 \Omega\xi + (4n\pi\Omega)^2 \cos^2 \Omega\xi \right\}. \end{aligned} \quad (16)$$

This equation reduces to $\langle |z(\xi)|^2 \rangle = \sin^2 \Omega\xi$ when the relations $\Delta_{mn} \ll \Delta \sin 2\theta$ and $\Omega \ll \alpha_n, \beta_n \ll 1$ are satisfied. This result agrees with Eq. (2) of Ref. [2] in the lowest order in Δ_{mn} with $2\bar{\omega} = n\omega$ and $n = 1$.

We studied the parametric resonance in neutrino oscillation under the inhomogeneous matter density. We expanded the matter density profile into a Fourier series and expressed the evolution equation of the neutrinos in a form of Hill's equation, which leads the oscillation probability to the parametric resonance. Noting that the resonance energy is controlled by the wavelength of the spatial variation of the matter density, we pointed out that the n -th Fourier mode of the matter density modifies the appearance probability through the resonance at the energy around the n -th dip. We verified the correspondence between the Fourier mode and the modification of the energy spectrum of the oscillation probability.

We further confirmed the parametric enhancement of the oscillation probability in a calculation where the matter density profile is periodically repeated. Unlike a simple parametric resonance, the enhancement turned out to be a slow oscillation, which respects the unitarity. We developed an analytic formula of the oscillation probability under the parametric enhancement. It well reproduced the frequencies of the slow- and the fast-oscillation of the numerical results. The agreement on the amplitude of the slow oscillation diminishes as the frequency of the slow oscillation gets larger.

The Fourier analysis of the parametric resonance cleared up the correspondence between the energy of the neutrinos and the scale of the spatial variation of the matter density. Below the energy of the MSW resonance,

a large-scale variation modifies the oscillation spectra in high energies while a small-scale one does in low energies. The effect of the density variation becomes large when neutrinos propagate through the core of the Earth. This effect should be kept under control in analyzing the oscillation experiments with very long baselines, including the next-generation accelerator experiments and the atmospheric neutrino experiments. Conversely, it can serve as a probe for tomography of the Earth [10] once the resonance effect is well understood. Systematic Fourier analyses on the parametric resonance will give a perspective to these experiments.

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